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LETTER TO THE EDITOR

Critical exponents of plane meanders

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Abstract. Meanders form a set of combinatorial problems concerned with the enumeration of self-avoiding loops crossing a line through a given number of points, n. Meanders are considered distinct up to any smooth deformation leaving the line fixed. We use a recently developed algorithm, based on transfer matrix methods, to enumerate plane meanders. This allows us to calculate the number of closed meanders up to $n = 48$, the number of open meanders up to $n = 43$, and the number of semi-meanders up to $n = 45$. The analysis of the series yields accurate estimates of both the critical point and critical exponent, and shows that a recent conjecture for the exact value of the semi-meander critical exponent is unlikely to be correct, while the conjectured exponent value for closed and open meanders is not inconsistent with the results from the analysis.

Meanders form a set of unsolved combinatorial problems concerned with the enumeration of self-avoiding loops crossing a line through a given number of points [1]. Meanders are considered distinct up to any smooth deformation leaving the line fixed. This problem seems to date back at least to the work of Poincaré on differential geometry [2]. Since then it has, from time to time, been studied by mathematicians in various contexts, such as the folding of a strip of stamps [3, 4] or the folding of maps [5]. More recently it has been related to enumerations of ovals in planar algebraic curves [6] and the classification of 3-manifolds [7]. During the last decade or so it has received considerable attention in other areas of science. In computer science, meanders are related to the sorting of Jordan sequences [8] and have been used for lower-bound arguments [9]. In physics, meanders are relevant to the study of compact foldings of polymers [10, 11], properties of the Temperley–Lieb algebra [12, 13], matrix models [14, 15], and models of low-dimensional gravity [16].

A *closed meander* of order n is a closed self-avoiding loop crossing an infinite line $2n$ times (see figure 1). The meandric number M_n is simply the number of such meanders distinct up to smooth transformations. Note that each meander forms a single connected loop. The number of closed meanders is expected to grow exponentially, with a sub-dominant term given by a critical exponent, $M_n \sim CR^{2n}/n^{\alpha}$. The exponential growth constant R is often called the *connective constant*. Thus the generating function is expected to behave as

$$\mathcal{M}(x) = \sum_{n=1}^{\infty} M_n x^n \sim A(x)(1 - R^2 x)^{\alpha-1} \tag{1}$$

and hence have a singularity at $x_c = 1/R^2$ with exponent $\alpha - 1$. The first meandric numbers are $M_1 = 1$, $M_2 = 2$ and $M_3 = 8$. One can extend the definition to *multi-component systems of closed meanders*, where we allow configurations with several disconnected closed loops. The

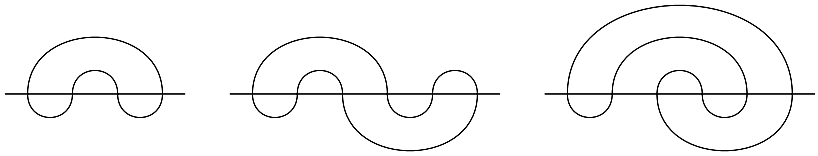

Figure 1. A few examples of closed meanders of order 2 and 3, respectively.

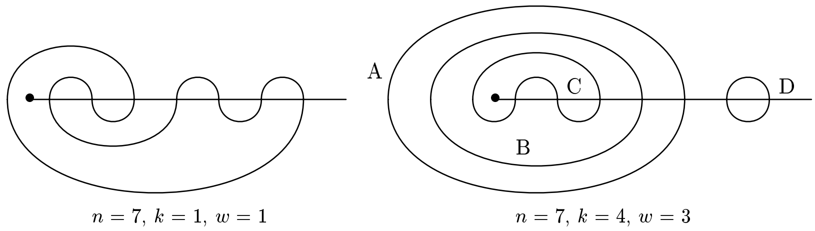

Figure 2. Two examples of semi-meanders. The first one has a single loop, winds around the origin once, and contains seven crossings. The second semi-meander has four loops (labelled A–D), winds around the origin three times, and again contains seven crossings.

meandric numbers $M_n^{(k)}$ are then the number of meanders with $2n$ crossings and k independent loops.

An *open meander* of order n is a self-avoiding curve running from west to east while crossing an infinite line n times. The number of such curves is m_n and we can define a generating function for this problem in analogy with (1). It should be noted [1] that $M_n = m_{2n-1}$, and hence the critical exponent is identical to that of closed meanders and the connective constant is R.

Finally, instead of looking at intersections with an infinite line one could consider a semi-infinite line and allow the curve to wind around the end-point of the line [10]. A *semi-meander* of order n is a closed self-avoiding loop crossing the semi-infinite line n times. The number of semi-meanders of order n is denoted by $\overline{M}_n \sim C'\overline{R}/n^{\overline{\alpha}}$ and we define a generating function as in (1). In this case a further interesting generalization is to study the number of semi-meanders $\overline{M}_n(w)$ which wind around the end-point of the line exactly w times. Again we could also study systems of multi-component semi-meanders according to the number of independent loops. Two semi-meanders are shown in figure 2.

In a recent paper it was argued that the meander problem is related to the gravitational version of a certain loop model [16]. From the conformal field theory of the model, conjectures were proposed for the exact critical exponent of closed and open meanders, $\alpha = (29 + \sqrt{145})/12 = 3.420\,1328\ldots$, as well as the exponent for semi-meanders, $\overline{\alpha} = 1 + \sqrt{11}(\sqrt{29} + \sqrt{5})/24 = 2.053\,1987\ldots$. This work has recently been extended to multi-component systems of closed and semi-meanders [17]. Conjectures were then given for the critical exponents as functions of the loop fugacity q. These were checked numerically [17] and found to be correct within numerical error. In this letter we analyse extended series for the meander generating functions. Using differential approximants we obtain accurate estimates for the exponents and find that the conjecture for the semi-meander exponent is unlikely to be correct, while the conjecture for closed meanders is not inconsistent with the results from the analysis.

The difficulty in the enumeration of most interesting combinatorial problems is that, computationally, they are of exponential complexity. Initial efforts at computer enumeration

Table 1. The number, M_n, of connected closed meanders with $2n$ crossings.

n	M_n	n	M_n	n	M_n
1	1	9	933 458	17	59 923 200 729 046
2	2	10	8 152 860	18	608 188 709 574 124
3	8	11	73 424 650	19	6 234 277 838 531 806
4	42	12	678 390 116	20	64 477 712 119 584 604
5	262	13	6 405 031 050	21	672 265 814 872 772 972
6	1 828	14	61 606 881 612	22	7 060 941 974 458 061 392
7	13 820	15	602 188 541 928	23	74 661 728 661 167 809 752
8	110 954	16	5969 806 669 034	24	794 337 831 754 564 188 184

Table 2. The number, m_n, of connected open meanders with n crossings.

n	m_n	n	m_n	n	m_n
1	1	16	252 939	31	5 969 806 669 034
2	1	17	933 458	32	15 012 865 733 351
3	2	18	2 172 830	33	59 923 200 729 046
4	3	19	8 152 860	34	151 622 652 413 194
5	8	20	19 304 190	35	608 188 709 574 124
6	14	21	73 424 650	36	1 547 365 078 534 578
7	42	22	176 343 390	37	6 234 277 838 531 806
8	81	23	678 390 116	38	15 939 972 379 349 178
9	262	24	1 649 008 456	39	64 477 712 119 584 604
10	538	25	6 405 031 050	40	165 597 452 660 771 610
11	1 828	26	15 730 575 554	41	672 265 814 872 772 972
12	3 926	27	61 606 881 612	42	1733 609 081 727 968 492
13	13 820	28	152 663 683 494	43	7060 941 974 458 059 344
14	30 694	29	602 188 541 928		
15	110 954	30	1503 962 954 930		

of meanders were based on direct counting. Lando and Zvonkin [1] studied closed meanders, open meanders and multi-component systems of closed meanders, while Di Francesco *et al* [11] studied semi-meanders. In this letter we use a new and improved algorithm [18], based on transfer matrix methods, to enumerate various meander problems such as closed, open and semi-meanders. The method is similar to the transfer matrix technique devised by Enting [19] in his pioneering work on the enumeration of self-avoiding polygons. The first terms in the series for the meander-generating function can be calculated using transfer matrix techniques. This involves drawing an intersection perpendicular to the infinite line. Meanders are enumerated by successive moves of the intersection, so that one crossing at a time is added to the meanders. A preliminary description of the algorithm can be found in [18] and further details will appear elsewhere. A very closely related algorithm was used and described in [17].

The enumerations undertaken thus far are too numerous to detail here. We only give the results for connected closed meanders M_n, open meanders m_n, and semi-meanders which wind around the origin any number of times and have only a single loop $\overline{M}_n$. The numbers of such meanders are listed in tables 1–3.

We analysed the series by the numerical method of differential approximants [20]. Estimates of the critical point and critical exponents were obtained by averaging values obtained from inhomogeneous differential approximants, chosen such that most, if not all, series terms were used. Some approximants were excluded from the averages because the estimates were obviously spurious. The error quoted for these estimates reflects the spread (basically, one

Table 3. The number, $\overline{M}_n$, of connected semi-meanders with n crossings.

n	$\overline{M}_n$	n	$\overline{M}_n$	n	$\overline{M}_n$
1	1	16	1 053 874	31	42 126 805 350 798
2	1	17	3 328 188	32	137 494 070 309 894
3	2	18	10 274 466	33	455 792 943 581 400
4	4	19	32 786 630	34	1 493 892 615 824 866
5	10	20	102 511 418	35	4 967 158 911 871 358
6	24	21	329 903 058	36	16 341 143 303 881 194
7	66	22	1 042 277 722	37	54 480 174 340 453 578
8	174	23	3 377 919 260	38	179 830 726 231 355 326
9	504	24	10 765 024 432	39	600 994 488 311 709 056
10	1 406	25	35 095 839 848	40	1 989 761 816 656 666 392
11	4 210	26	112 670 468 128	41	6 664 356 253 639 465 480
12	12 198	27	369 192 702 554	42	22 124 273 546 267 785 420
13	37 378	28	1 192 724 674 590	43	74 248 957 195 109 578 520
14	111 278	29	3 925 446 804 750	44	247 100 408 917 982 623 532
15	346 846	30	12 750 985 286 162	45	830 776 205 506 531 894 760

Table 4. Estimates of the critical points and exponents of the meander-generating functions for closed, open and semi-meanders, as obtained from second-order differential approximants. L is the degree or the inhomogeneous polynomial.

	Closed meanders		Open meanders		Semi-meanders	
L	x_c	$\alpha - 1$	x_c	$\alpha - 1$	x_c	$\overline{\alpha} - 1$
0	0.081 546 71(24)	2.421 04(42)	0.285 563 61(40)	2.421 29(36)	0.285 564 437(10)	1.053 693(12)
1	0.081 546 84(14)	2.420 84(33)	0.285 564 16(19)	2.420 75(20)	0.285 564 48(10)	1.053 62(16)
2	0.081 546 912(59)	2.420 79(45)	0.285 564 18(64)	2.420 72(63)	0.285 564 47(13)	1.053 58(27)
3	0.081 546 916(84)	2.420 69(17)	0.285 563 86(33)	2.421 09(34)	0.285 564 436(31)	1.053 693(47)
4	0.081 546 950(46)	2.420 61(10)	0.285 563 90(50)	2.421 01(46)	0.285 564 433(29)	1.053 700(34)
5	0.081 546 901(82)	2.420 74(18)	0.285 564 06(10)	2.420 88(13)	0.285 564 437(24)	1.053 692(32)
6	0.081 546 917(67)	2.420 65(21)	0.285 563 94(28)	2.421 01(32)	0.285 564 425(65)	1.053 699(96)
7	0.081 546 910(72)	2.420 70(21)	0.285 564 07(10)	2.420 88(12)	0.285 564 413(58)	1.053 717(71)
8	0.081 546 82(16)	2.420 90(30)	0.285 564 08(10)	2.420 87(13)	0.285 564 425(46)	1.053 706(54)
9	0.081 546 68(32)	2.421 15(60)	0.285 564 096(83)	2.420 83(11)	0.285 564 434(52)	1.053 692(79)
10	0.081 546 71(26)	2.421 07(45)	0.285 564 14(16)	2.420 78(18)	0.285 564 433(47)	1.053 695(63)

standard deviation) among the approximants. Note that these error bounds should *not* be viewed as a measure of the true error as they cannot include possible systematic sources of error. In table 4 we have listed the results from this analysis. Our first observation is that the critical points of open and semi-meanders are identical, and thus so are the connective constants R for all the problems (recall that, for closed meanders, $x_c = 1/R^2$). Clearly the most accurate estimates are obtained from the semi-meander series and from this we estimate, conservatively, that $x_c = 0.285\,5644(2)$ and thus $R = 3.501\,837(3)$. Secondly, as expected, open and closed meanders have the same critical exponent, which we estimate to be $\alpha = 3.4208(6)$. This could, though only marginally, be consistent with the conjectured value $\alpha = (29 + \sqrt{145})/12 = 3.420\,1328\ldots$. For semi-meanders we estimate $\overline{\alpha} = 2.0537(2)$, which is not consistent with the conjecture $\overline{\alpha} = 1 + \sqrt{11}(\sqrt{29} + \sqrt{5})/24 = 2.053\,1987\ldots$.

In order to gain a better understanding of the behaviour of the exponent estimates it is useful to plot them against the number of terms used to form the differential approximant. In particular we can check whether or not the estimates asymptote or whether they are drifting with

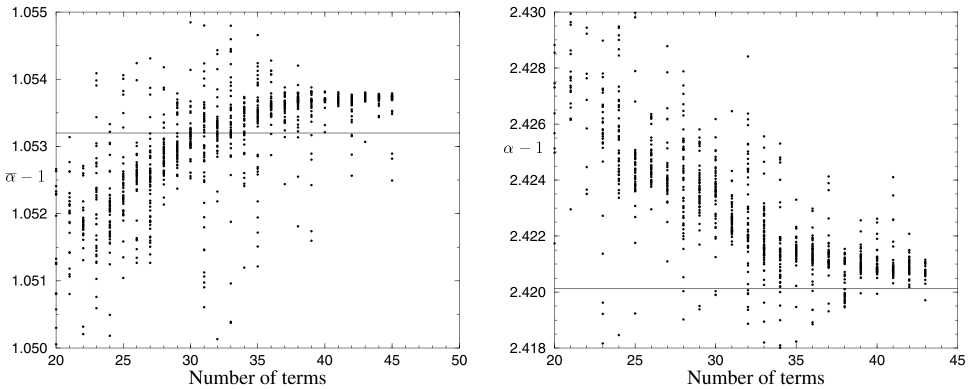

Figure 3. Estimates of the critical exponents of the semi-meander-generating function, $\overline{\alpha}-1$, and the open meander-generating function, $\alpha-1$, versus the number of terms from the series used by the differential approximants. Each point represents an estimate obtained from a particular second-order differential approximant.

the length of the series. In figure 3 we have done this for semi-meanders and open meanders. These plots strongly reinforce the comments made above. The exponent estimates for semi-meanders increase as more terms are used and appear to settle down to an asymptotic value above the conjectured value. For open meanders the exponent estimates decrease and approach the conjectured value as more terms are used. It is quite likely that, with a longer series, the estimates would actually converge to the conjectured value, though it is also possible that the estimates could settle at a value just above the conjectured value. If we look at the estimates in table 4 we note that, for both open and closed meanders, the exponent estimates decrease as the critical point estimates increase. It is possible that as x_c approaches the estimate obtained from semi-meanders the exponent estimates approach the conjectured value. To check this we plotted (in figure 4) the exponent estimates versus the critical point estimates for open and closed meanders. The solid lines are the conjectured exponent value and the best estimate for the critical point based on the semi-meander analysis. Clearly the estimates pass extremely close to the intersection between the solid lines, lending further support to the possibility that the conjecture for α is correct.

Next, we looked for non-physical singularities and found that both the open and semi-meander-generating functions have a singularity at $-1/R$ with an exponent whose value is consistent with $\alpha-1$. These generating functions also have a pair of singularities in the complex plane at $\pm 0.685(5)$i. The exponent estimates are quite poor, but consistent with the value $\alpha-1$.

Finally, we turned our attention to the 'fine structure' of the asymptotic behaviour of the meandric numbers,

$$M_n \sim R^{2n} \sum_{i=0} c_i / n^{\alpha+f(i)} \tag{2}$$

$$m_n \sim R^n \sum_{i=0} [c_i / n^{\alpha+f(i)} + (-1)^n d_i / n^{\alpha+f(i)}] \tag{3}$$

$$\overline{M}_n \sim R^n \sum_{i=0} [c_i / n^{\overline{\alpha}+f(i)} + (-1)^n d_i / n^{\alpha+f(i)}]. \tag{4}$$

The alternating sign terms are due to the singularity at $-1/R$. Fitting the meandric numbers to these formulae we found excellent convergence when $f(i) = i$. This corresponds to the

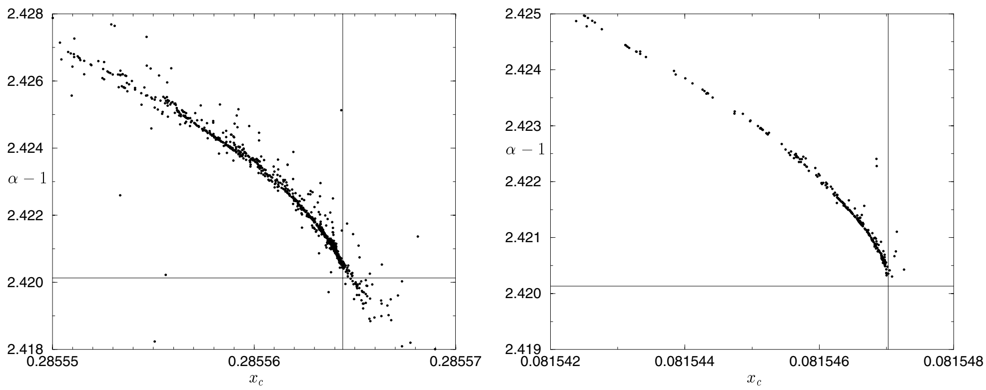

Figure 4. Estimates of the critical exponents for open (left panel) and closed meanders versus the corresponding critical point estimates as obtained from differential approximants.

case where there are only analytic corrections-to-scaling terms. The leading amplitudes c_0 are of special interest, and for closed, open and semi-meanders we found the values 0.339(1), 11.45(3), and 0.688(1), respectively.

E-mail or WWW retrieval of series.

The series for the various generating functions studied in this letter can be obtained via e-mail by sending a request to `I.Jensen@ms.unimelb.edu.au` or via the World Wide Web at the URL `http://www.ms.unimelb.edu.au/~iwan/` by following the instructions.

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